ALGEBRA II — Lesson 0118 - Compound Interest

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1. Interest

If one person borrows money from another, they may pay *interest* for the time that the money is borrowed. This is money given by the borrower to the lender.

1.1. **Simple Interest.** *Simple interest* is computed when the loan is repaid. It consists of multiplying an interest rate times the time.

Let A_0 denote the amount loaned and let r denoted the interest rate. The additional amount repaid after t time periods is A_0rt, so the total amount repaid is

$$A_t = A_0(1 + rt).$$

Example 1. Suppose John borrows $A_0 = 1000$ dollars, and keeps it for $t = 5$ years at a simple interest rate of 5% per year. How much does John owe after ten years?

Solution. After 10 years, John will repay Ted

$$A_{10} = A_0(1 + rt) = 1000(1 + (0.05) \times 10) = 1000(1.5) = 1500.$$

So, John owes $1500 after ten years. □

1.2. **Compound Interest.** Suppose we invest 1000 dollars at an interest rate of 10 percent compounded annually. The amount we have invested remains the same until one year passes, at which point 10 percent of the amount is added to the total. If we let A_t denote the amount invested after t years, then

- $A_0 = 1000$
- $A_1 = 1000 + (0.1)1000 = 1100$
- $A_2 = 1100 + (0.1)1100 = 1210$
- $A_3 = 1210 + (0.1)1210 = 1331$

We see that the rate at which this grows increases year by year; but the pattern is obscure. It is actually easier to see the pattern if we think more generally.

Let r be the annual interest rate, A_0 the initial investment, and A_t the amount after t years. Then

- $A_1 = A_0 + rA_0 = A_0(1 + r)$
- $A_2 = A_1 + rA_1 = A_1(1 + r) = A_0(1 + r)^2$
- $A_3 = A_2 + rA_2 = A_2(1 + r) = A_0(1 + r)^3$
- $A_t = A_0(1 + r)^t$

1.3. **Periodic Compound Interest.** Suppose that, instead of compounding annually, we compound quarterly; that is, every three months, or four times per year. Then, the periodic interest rate is the annual rate divided by four.

- $A_{1/4} = A_0 + (\frac{r}{4})A_0 = A_0(1 + \frac{r}{4})$
- $A_{1/2} = A_{1/4} + (\frac{r}{4})A_{1/4} = A_{1/4}(1 + \frac{r}{4}) = A_0(1 + \frac{r}{4})^2$
- $A_1 = A_0(1 + \frac{r}{4})^4$
- $A_t = A_0(1 + \frac{r}{4})^{4t}$

Generalize this further; let k denote the number of periods per year, so that we compound k times per year. Then, there are k times every year when we the amount in the account by $(1 + \frac{r}{k})$; these gives

$$A_t = A_0\left(1 + \frac{r}{k}\right)^{kt},$$

where r is the annual rate, k is the number of periods per year, and A_t is the amount after t years.

The more periods per year, the faster the amount grows, as this table demonstrates. We let the annual rate r be ten percent and the initial investment A_0 be one thousand. We compute the amount after five years for various values of k, to the nearest dollar:

k	A_0	A_1	A_2	A_3	A_4	A_5
1	1000	1100	1210	1331	1464	1611
2	1000	1103	1216	1340	1477	1629
4	1000	1104	1218	1345	1485	1639
12	1000	1105	1220	1348	1489	1645
365	1000	1105	1221	1350	1492	1649
8760	1000	1105	1221	1350	1492	1649

This table demonstrates two facts:

- as k increases, the investment grows faster;
- as k increases, the rate at which the investment grows faster slows down.

Example 2. Harold invests 7625 dollars in the bank at an annual interest rate of 2.5%, compounded monthly. How much money is this account worth after seven years?

Solution. We have $A_0 = 7625$, $r = 0.025$, $k = 12$, and $t = 7$. This gives

$$A_7 = A_0\left(1 + \frac{r}{k}\right)^{kt} = 7625\left(1 + \frac{0.025}{12}\right)^{12\cdot 7} = 9081.60.$$

So, after seven years, Harold has \$ 9081.60. □